Liquid Crystals
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713926090

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Online publication date: 06 August 2010

To cite this Article Casquilho, J. P.(1999) 'Linear analysis of pattern formation in nematics in oblique magnetic fields', Liquid Crystals, 26: 4, 517 — 524

To link to this Article: DOI: 10.1080/026782999204958

URL: http://dx.doi.org/10.1080/026782999204958

LIQUID CRYSTALS, 1999, VOL. 26, NO. 4, 517–524

Linear analysis of pattern formation in nematics in oblique magnetic fields

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(*Received 7 July 1998; in final form 21 October 1998; accepted 2 November 1998*)

The dynamics of nematic director field reorientation in non-Fréedericksz geometries, after a magnetic field **H** is applied at an oblique angle relatively to the initial homogeneous director $\mathbf{n}_0$ (**H** not normal to $\mathbf{n}_0$), is studied considering a magnetic reorientation driven by hydrodynamic instabilities (with backflow). This study is carried out for bounded samples between two parallel plates with planar boundary conditions and with rigid anchoring. Linear stability and wave vector selection analysis predict that, when the angle of the magnetic to the initial director field is increased, for a given magnetic field intensity, two transitions from a homogeneous to a transient distorted director field reorientation can occur: a transition at a first critical angle to an aperiodic distorted director field and a transition at a second critical angle to a periodic distorted director field. It is shown that the periodic mode is cut off at a higher reduced field when the magnetic field acts away from the normal direction.

1. Introduction

The dynamics of nematic liquid crystals under external fields provide fascinating examples of non-linear phenomena in far from equilibrium systems. The formation of ordered patterns in samples as a transient response to a destabilizing magnetic and/or electric field is one striking feature [1, 2]. The study of the external field induced instabilities in nematic liquid crystals is of great importance for the operation of many liquid crystal displays. In this work the possibility of the formation of a transient periodic distortion in the director field induced by an applied oblique magnetic field in non-Fréedericksz geometries is investigated theoretically using a dynamical stability analysis. This study will be presented here with the help of the Ericksen–Leslie nematic hydrodynamic theory [1, 2].

When a magnetic field **H** is applied at an angle α with respect to the homogeneous director field $\mathbf{n}_0$ of a nematic liquid crystal monodomain (**H** not normal to $\mathbf{n}_0$, as shown in figure 1) the director field reorients towards the magnetic field to an equilibrium configuration determined by the balance between the magnetic and the elastic torques acting on it. There is experimental evidence that this magnetic reorientation of the director can be either uniform or inhomogeneous in space, depending on the magnitude of the angle α [3–8]. In the last case the reorientation of the director may result in a transient periodic pattern, giving rise to striped textures, and

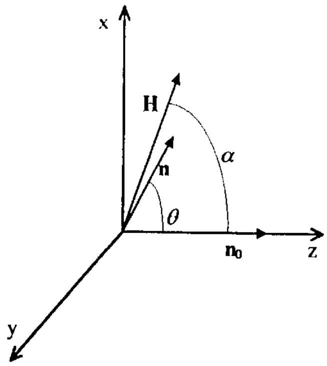

Figure 1. Definition of the sample geometry. The director and the magnetic fields are in the (xz) plane. Homogeneous initial orientation of the director: $\mathbf{n}_0 = (0, 0, 1)$. Magnetic field: $\mathbf{H} = (H \sin \alpha, 0, H \cos \alpha)$. Perturbed director: $\mathbf{n} = (\sin \theta, 0, \cos \theta)$. The sample is bounded in the OY direction; the plates are parallel to the (xz) plane and cut the OY axis at $y = \pm d/2$.

an induced backflow should then be involved [4–6]. Accordingly, a non-zero velocity field $\boldsymbol{v}$ has to be considered in the general study of the dynamics of this magnetic reorientation, as shown by several studies of the magnetic and/or electric field induced instabilities in Fréedericksz geometries [9–16]. Although the subject of the external field induced instabilities in liquid crystals continues to receive attention from scientists working in this area since the pioneering work of Brochard and co-workers [17], the dynamical problem in non-Fréedericksz geometries, i.e. where **H** is taken at an

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Journal of Liquid Crystals ISSN 0267-8292 print/ISSN 1366-5855 online ©1999 Taylor & Francis Ltd
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arbitrary angle with respect to $\mathbf{n}_0$, is poorly understood. Following the study of Karn and co-workers on the bistability and dynamical response (neglecting backflow) of a nematic cell induced by the sudden rotation of an applied magnetic field [18], Kini [19] approached the problem in the case where $\mathbf{H}$ is slowly rotated away from $\mathbf{n}_0$ (for nematics with positive anisotropy of the magnetic susceptibility χ_a), in which case it can be treated under statics by neglecting transient effects. A dynamical analysis was then attempted by studying the stability of the static solutions, corresponding to homogeneously deformed configurations, but no transient periodic structure was reported. Experimentally, in order to get the striped texture, the magnetic field is suddenly applied at (or rotated by) an angle α which has to exceed a critical value in the case $\chi_a > 0$ [6] or be smaller than a critical value in the case $\chi_a < 0$ [4]. In a static analysis [20, 21] the minimization of an ansatz based distortion Frank free energy could explain the existence of such a critical angle separating the uniform director magnetic reorientation from the periodically distorted director reorientation. Here, the corresponding dynamical linear analysis is attempted.

The goal of this work is to study an instability driven magnetic reorientation in the bulk of a nematic monodomain following the sudden application of an oblique magnetic field at an arbitrary angle α with respect to the unperturbed director (or equivalently, the previously aligned sample is suddenly rotated with respect to the magnetic field, as in the NMR experiments with polymer liquid crystals (PLC) reported in [3, 5–8]). Backflow is taken into account. The study considers bounded samples between two parallel plates with planar boundary conditions and rigid anchoring, corresponding to the twist Fréedericksz transition geometry when $\alpha = 90°$. The study is carried out for nematics with $\chi_a > 0$. A dynamical linear stability analysis based on a two dimensional director field predicts two transitions from the homogeneous to the distorted director field reorientation when the angle α is increased: a transition at a first critical angle α_{c1} to a non-periodic distorted state and a transition at a second critical angle $\alpha_{c2} > \alpha_{c1}$ to a periodic distorted director. The existence of this second critical angle, separating the aperiodical from the periodically distorted states, was envisaged by Kini [14] in the context of results obtained in the study of magnetic field induced transient periodic structures in a Fréedericksz geometry. In this work a variational method is employed that allows it to be shown that α_{c1} and α_{c2} are dependent on the applied magnetic field, the sample thickness, the magnetic parameter χ_a and a Frank elastic constant, and that α_{c2} is also dependent on several Leslie and Frank viscoelastic parameters. The limitations of this mathematical model are discussed. The results show that a two dimensional director field is sufficient to predict that the periodic mode will be cut off at a higher reduced field when the magnetic field acts away from the normal direction. Critical sample thicknesses d_{c1} and d_{c2} are put in evidence for the aperiodic and the periodic deformed director reorientations, respectively. It is shown that both critical thicknesses depend on the magnetic coherence length and that d_{c2} is also dependent on several viscoelastic parameters.

2. Mathematical model

Consider a bulk nematic aligned monodomain. A magnetic field $\mathbf{H} = (H \sin \alpha, 0, H \cos \alpha)$ is applied at an angle α with respect to the initial homogeneous director $\mathbf{n}_0 = (0, 0, 1)$, as shown in figure 1. The dynamics of the nematic director field will be studied using the Ericksen–Leslie equations [1]. To study this dynamics in the case of a twist–bend instability, the following general velocity and director fields are considered:

$$v_x(y, z, t), \quad v_y = v_z = 0 \tag{1a}$$

$$n_x = \sin \theta(y, z, t), \quad n_y = 0, \quad n_z = \cos \theta(y, z, t) \tag{1b}$$

which obey the usual constraints of incompressibility of the fluid and unit vector $\mathbf{n}$.

In order to study the transition from the homogeneous to the instability driven reorientation of the director field, the stability of the uniform reorientation with respect to the development of a twist–bend deformation will be analysed. Standard linear stability analysis will be followed, valid near the transition point. Therefore, for the velocity and director fields (1) the following functions will be taken

$$\begin{aligned} v_x(y, z, t) &= 0 + \xi_v(y, z, t) \\ \theta(y, z, t) &= u(t) + \xi_\theta(y, z, t) \end{aligned} \tag{2}$$

where the unperturbed solution $u(t)$ corresponds to the uniform reorientation, and with the perturbations of the velocity and the director fields given respectively by

$$\begin{aligned} \xi_v(y, z, t) &\equiv v_x = v_0(t) \cos(q_y y) \sin(q_z z) \\ \xi_\theta(y, z, t) &= \theta_0(t) \cos(q_y y) \cos(q_z z) \end{aligned} \tag{3}$$

representing a twist–bend distortion in the harmonic approximation. The ansatz is consistent with the planar boundary conditions at $y = \pm d/2$ of the twist Fréedericksz geometry, with $q_y = \pi/d$ where d is the sample thickness in the OY direction [10].

The procedure adopted is as follows. In a first step, the fields (1) are inserted in the general dynamic equations, from which there result two coupled nonlinear equations for θ and v_x. After inserting the fields (2) in those equations, following standard stability analysis [22], one obtains the variational equations up to first order in the perturbations ξ_v and ξ_θ The terms that cancel out in the variational equations correspond to the uniform reorientation equation [1]:

$$\gamma_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\frac{1}{2}\chi_a H^2 \sin 2[u(t) - \alpha] \tag{4}$$

Finally, the linearized equations for the perturbations around the unperturbed state, taken at the instant of applying the field, i.e. at $u(t=0)=0$, and using (3) read

$$\rho \frac{\mathrm{d}v_0}{\mathrm{d}t} = -(\eta_a q_y^2 + \eta_c q_z^2) v_0 - \alpha_2 q_z \frac{\mathrm{d}\theta_0}{\mathrm{d}t} \tag{5}$$

$$\gamma_1 \frac{\mathrm{d}\theta_0}{\mathrm{d}t} = -\alpha_2 q_z v_0 - a\theta_0 \equiv F_\theta \tag{6}$$

where

$$a = K_2 q_y^2 + K_3 q_z^2 + \chi_a H^2 \cos 2\alpha. \tag{7}$$

The viscoelastic parameters used in this work are the Leslie viscosity coefficients α_i, $i=1, \ldots, 6$, the Miesowicz viscosities defined by $\eta_a = \alpha_4/2$, $\eta_b = (\alpha_3 + \alpha_4 + \alpha_6)/2$ and $\eta_c = (\alpha_4 + \alpha_5 - \alpha_2)/2$, and the splay, twist and bend Frank elastic constants K_i, $i = 1, 2, 3$, respectively. While some of the Leslie viscosities are or may be negative, the Miesowicz viscosities are positive parameters of the materials [1]. γ_1 is the rotational viscosity that is a positive quantity defined by $\gamma_1 = \alpha_3 - \alpha_2$ and is the effective viscosity η_{twist} associated with a pure twist mode [1]. Only five of the viscosity coefficients are independent parameters [1].

To perform a stability analysis (Appendix), this system is put in canonical form with the substitution of equation (6) in (5), from which there results the equivalent system consisting of equation (6) and the following equation

$$\rho \frac{\mathrm{d}v_0}{\mathrm{d}t} = -c v_0 + \frac{\alpha_2}{\gamma_1} q_z a \theta_0 \equiv F_v \tag{8}$$

where

$$c = \eta_a q_y^2 + \frac{\eta_c \eta_{\mathrm{bend}}}{\gamma_1} q_z^2 \tag{9}$$

where η_{bend} is the effective viscosity associated with a pure bend mode [1], and is a positive quantity defined by $\eta_{\mathrm{bend}} = \gamma_1 - \alpha_2^2/\eta_c$.

2.1. *Limitations of the mathematical model*

The method of the variational equations followed here allowed the linear dynamic equations (6) and (8) for the perturbations of the director field ξ_θ and the velocity field ξ_v, respectively, to be obtained. Attention must be given now to the equation for $u(t)$. The requirement that $u(t)$ should vanish at $t=0$ allows equation (4) to be linearized:

$$\gamma_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + \Gamma u(t) = \frac{1}{2}\chi_a H^2 \sin 2\alpha \tag{10}$$

where

$$\Gamma = \chi_a H^2 \cos 2\alpha. \tag{11}$$

Equation (10) has the solution

$$u(t) = \frac{\sin 2\alpha}{2\cos 2\alpha}[1 - \exp(-t\Gamma/\gamma_1)]. \tag{12}$$

This result shows that $u(t)$ diverges at $\alpha = \pi/4$. It also shows that, in the limit $t=0$, $u(t)$ actually becomes indeterminate when $\alpha = \pi/4$, which implies by equation (2) that the director defining angle $\theta(t)$ is also indeterminate in the same limit. Equation (12) also shows that $u(t)$ is damped for $\alpha < \pi/4$. The reason for this is that, in the linear limit, the total magnetic torque becomes split into two parts. With the definition of $u(t)$ as in equations (2), (4), (10) and (12), the part $(1/2)\chi_a H^2 \sin 2\alpha$ is removed from the linear equations for the perturbations. The remaining torque term (11) has the property that it has a destabilizing effect for $\alpha > \pi/4$ and a stabilizing effect for $\alpha < \pi/4$. So the physical problem arising from equation (12) when $\alpha \leqslant \pi/4$ is due to the fact that this term does not fully represent the magnetic torque. This means that the results obtained via the equations (6, 8) should only be correct for values of α not too far from the Fréedericksz geometries, the approximation becoming poorer when α approaches $\pi/4$. To study the whole range $0 < \alpha < \pi/2$, the solution of the coupled non-linear equations for θ and v_x is then required.

3. Wave vector selection and transition to distorted state

The wavevector $\mathbf{q}$ of the distortion is an internal parameter of the system. In order to study the stability of the system with respect to the control parameters, which are the parameters of the material (the viscoelastic parameters and the anisotropy of the magnetic susceptibility) and the external parameters (sample dimension d, magnetic field H and angle α), one should eliminate the parameter $\mathbf{q}$ by seeking its selected value. The wave vector is selected during the initial steps of the magnetic reorientation. Accordingly, the method followed here will be a standard linear analysis, valid near the transition. This method assumes that there is no interaction between modes, which implies that the selected wave vector will correspond to the fastest growing mode of the instability.

Consider a bounded sample in the OY direction with thickness d. For planar boundary conditions, corresponding to the twist Fréedericksz geometry when $\alpha = 90°$, one has [10]

$$q_y = \pi/d \tag{13}$$

In the linear theory, the selected wave vector will be the value of q_z that corresponds to the maximum growth rate of the instability. For the system (6, 8) the growth rate is the eigenvalue λ_+ given by equation (A2). The plot of $\lambda_+(q_z)$ with the angle α as a control parameter, for a given magnetic field, is shown in figure 2, for the viscoelastic parameters of PAA in the table. This shows the following behaviour: the system becomes unstable at a critical angle $\alpha_{c1} \cong 46°$, obtained by inspection of the curves of λ_+. But above this transition, for a certain range of values of the angle α, the maximum growth rate is kept at $q_z = 0$. Therefore here the fastest growth rate corresponds to an aperiodic distortion. A second transition happens at another critical value α_{c2}, when $q_z(\lambda_{\max})$ goes continuously from zero to non-zero values. This is a transition to a periodic state and is a dynamical equivalent of a second order phase transition, where the wavevector $q_z(\lambda_{\max})$ plays the role of the order parameter [23]. This second critical angle can be obtained by solving in order to α the equation defining the second transition, which is for the non-zero selected wave vector

$$q_z(\lambda_{\max}) = 0. \tag{14}$$

In consequence, one seeks the solutions of equation

$$\partial\lambda_+/\partial q_z = 0. \tag{15}$$

One solution of equation (15) is

$$q_z = 0. \tag{16}$$

Solving equation (A7) for $\cos 2\alpha_c$ and inserting (13) and (16) in it yields for the first critical angle

$$\cos 2\alpha_{c1} = -K_2(\pi/d)^2/\chi_a H^2 \tag{17a}$$

or

$$\cos 2\alpha_{c1} = -\left(\frac{H^*}{H}\right)^2 \tag{17b}$$

where

$$H^* = (K_2\pi^2/\chi_a d^2)^{1/2} \tag{18}$$

is the critical field for the aperiodic twist Fréedericksz transition [1]. This shows that a field $H > H^*$ is required. The resolution of equation (17) for the values of PAA in the table gives $\alpha_{c1} = 46°$, in agreement with the graphical result.

In order to get an analytical solution for α_{c2} in terms of the material and other control parameters, one seeks to solve equation (14) starting with a simpler expression for the growth rate than λ_+ given by (A2). For this, following [9–14], the inertial term will be dropped in equation (8). This procedure is equivalent to the adiabatic approximation [24], which consists of looking at the system (6, 8) as consisting of equations of motion for two variables, one of which is 'slaved' by the other. In the physical problem under study, since the velocity results from a backflow driven by the (magnetic) reorientation of the director, one can say that the system is damped for $d\theta_0/dt = 0$. On account of this, one may solve equation (8) approximately by putting $dv_0/dt = 0$ and solving in order to v_0:

$$v_0 = \frac{\alpha_2}{\gamma_1}\frac{a}{c} q_z \theta_0. \tag{19}$$

This means that $v_0(t)$ immediately follows θ_0 and in this sense one can say that v_0 is 'slaved' by θ_0. The substitution of v_0 given by (19) in equation (6) results in the following equation for θ_0:

$$\gamma_{\mathrm{ef}} \frac{d\theta_0}{dt} = -a\theta_0 \tag{20}$$

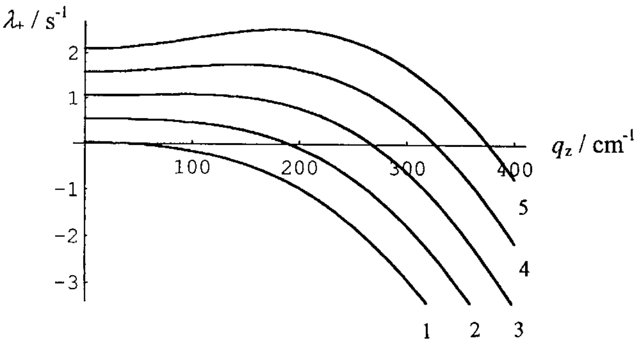

Figure 2. Plot of λ_+ given by equation (12) as a function of q_z (parameters of PAA with $\chi_a H^2 = 1$ erg cm^{-3} and with $q_y = \pi \times 10^2$ cm^{-1}). (1) $\alpha = 46°$; (2) $\alpha = 47°$; (3) $\alpha = 48°$; (4) $\alpha = 49°$; (5) $\alpha = 50°$. The system becomes unstable at $\alpha_{c1} \cong 46°$, but the periodic distortion is only selected at $\alpha_{c2} \cong 47°$.

Table Parameters used in the numerical simulations. α_i in g cm^{-1} s^{-1}; K_i in 10^{-7} dyn.

PAA		PBG [12]
$\alpha_1 = 0.043$	[2]	$\alpha_1 = -36.7$
$\alpha_2 = -0.069$	[2]	$\alpha_2 = -69.2$
$\alpha_3 = -0.002$	[2]	$\alpha_3 = 0.20$
$\alpha_4 = 0.068$	[2]	$\alpha_4 = 3.48$
$\alpha_5 = 0.047$	[2]	$\alpha_5 = 66.1$
$K_1 = 4.8$	[1]	$K_1 = 12.1$
$K_2 = 3.4$	[1]	$K_2 = 0.78$
$K_3 = 9.9$	[1]	$K_3 = 7.63$

with a given by equation (7) and where γ_{ef} is an effective viscosity given by

$$\gamma_{ef} = \gamma_1 - \frac{\alpha_2^2}{\eta_c + \eta_a(q_y^2/q_z^2)}. \tag{21}$$

The substitution of the ansatz (A1) for θ_0 in equation (20) yields for the growth rate s

$$s(\mathbf{q}) = -a/\gamma_{ef}. \tag{22}$$

This growth rate leads to the same bifurcation set as defined by equation (A7) (which is the interesting set in the case of real wave vectors as discussed in the Appendix) and is much simpler to manipulate algebraically than (A2). Expression (22) with $\alpha = 90°$ in (7) reduces to the one obtained by [10] in the study of the periodic twist Fréedericksz transition. A numerical plot of the growth rates defined by λ_+ and s as a function of q_z for PAA and with $\rho = 1\,\mathrm{g\,cm^{-3}}$ shows an excellent agreement between the two curves, except in the case of large samples and small values of q_z. In this case, the inertial term cannot be neglected. In the limit $d \to \infty$ and $q_z \to 0$ the adiabatic approximation breaks down, since it is the inertial term that prevents the divergence of v_0 predicted by equation (19).

To determine the non-zero value of the selected wave vector in the adiabatic approximation, one solves in order to q_z^2 the equation

$$\partial s/\partial q_z^2 = 0. \tag{23}$$

Scaling the solution of this equation by the sample dimension, one gets

$$\left(\frac{q_z}{q_y}\right)^2 = \frac{-\eta_a\eta_c\gamma_1K_3 + \{\alpha_2^2\eta_a\eta_cK_3 \times[\eta_a\gamma_1K_3 - \eta_c\eta_{bend}K_2(1+h^2\cos 2\alpha)]\}^{1/2}}{\eta_c^2\eta_{bend}K_3} \tag{24}$$

where $h = H/H^*$ is the reduced field with H^* given by equation (18). Figure 3 shows that the periodic mode is cut off at a higher reduced field when the angle α goes from 90° to lower values. Solving equation (24) at the transition point $[q_z(s_{max}) = 0]$ gives the relation

$$-h^2\cos 2\alpha_{c2} = 1 + \frac{\eta_a\gamma_1}{\alpha_2^2}\frac{K_3}{K_2}. \tag{25}$$

This equation shows that, if h is fixed, a critical value of α can be found below which the periodic pattern may not appear, and *vice-versa*.

3.1. *Critical angle*

Solving equation (25) in order to the angle α and using (13) and (18) yields

$$\cos 2\alpha_{c2} = -\left(\frac{H^*}{H}\right)^2\left(1 + \frac{\eta_a\gamma_1}{\alpha_2^2}\frac{K_3}{K_2}\right) \tag{26}$$

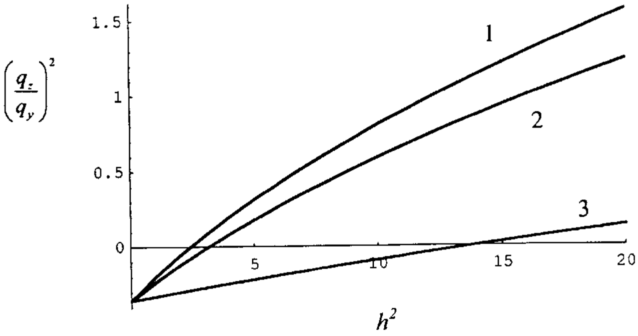

Figure 3. Plot of the square of the reduced wave vector given by equation (24) as a function of the square of the reduced magnetic field, for the same parameters as for figure 2. (1) $\alpha = 90°$; (2) $\alpha = 70°$; (3) $\alpha = 50°$. This shows that the periodic mode becomes cut off at a higher reduced field when the angle α goes from 90° to lower values.

Equations (17 *b*) and (26) show that the second critical angle $\alpha_{c2} > \alpha_{c1}$ and depends on three ratios of the control parameters: the ratio of the critical aperiodic Fréedericksz field to the applied field as in the first transition, the viscosity ratio $\eta_a\gamma_1/\alpha_2^2$ and the elastic ratio K_3/K_2. One can see that for materials where the product of these two viscoelastic ratios departs significantly from zero, the second critical angle can be distinctly different from the first critical angle. Such a case is shown in figure 4 for PAA, a low molecular mass liquid crystal (LMWLC) for which the above product of viscoelastic ratios if 1.46. Also in figure 4 is shown the case of PBG, a polymer liquid crystal (PLC) with viscoelastic parameters also shown in the table, and for which the

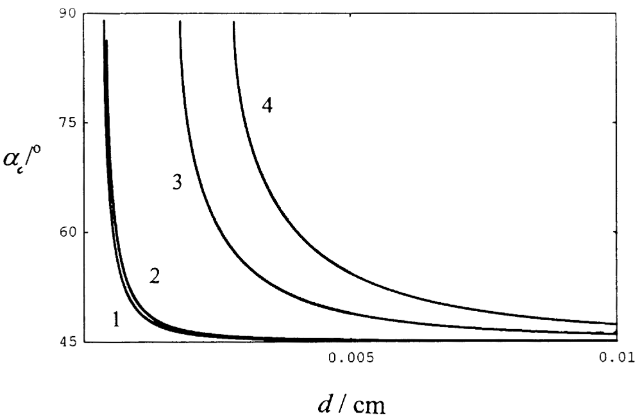

Figure 4. Critical angles as a function of the sample dimension. These curves provide evidence for the existence of critical sample dimensions related to both the aperiodic and the periodic transitions (see text). (1) α_{c1} PBG; (2) α_{c2} PBG ($d_{c1} \cong d_{c2} = 4\,\mu\mathrm{m}$) (with $\chi_a H^2 = 5\,\mathrm{erg\,cm^{-3}}$); (3) α_{c1} PAA ($d_{c1} = 18\,\mu\mathrm{m}$); (4) α_{c2} PAA ($d_{c2} = 28\,\mu\mathrm{m}$) (with $\chi_a H^2 = 1\,\mathrm{erg\,cm^{-3}}$).

product of the two ratios is 0.25 and as a consequence the curves for both critical angles nearly collapse into the same curve.

3.2. *Critical sample thickness*

Equations (17) and (26) make evident critical values d_c for the sample thickness for a given field, below which there is no critical angle $\alpha_c \leqslant 90°$, as shown in figure 4. From (17) we get the first critical sample thickness d_{c1} that separates the homogeneous reorientation from the aperiodic reorientation

$$d_{c1} = \pi \xi_2 \tag{27}$$

where ξ_2 is the twist magnetic coherence length [1] defined by $\xi_2^2 = K_2 / \chi_a H^2$.

From equations (26) and (27) we get the second critical sample thickness d_{c2}, which separates the aperiodic from the periodic distorted reorientation

$$d_{c2} = d_{c1} \left(1 + \frac{\eta_a \gamma_1}{\alpha_2^2} \frac{K_3}{K_2} \right)^{1/2}. \tag{28}$$

The existence of a critical sample thickness for the development of a periodic pattern shows up in experimental results for the dynamics of a nematic PLC in a Fréedericksz geometry [16], where it was observed that in thin cells the periodic reorientation is replaced by a uniform reorientation. Moreover, the observed periodic variation of the director in [16] is two dimensional, which supports the single distortion angle description of the director used in this work.

To conclude this section, the results obtained from the preceding results by putting $d \rightarrow \infty$ or $q_y = 0$ are analysed. This case corresponds to a pure bend distortion in an unbounded sample. In this case, both transitions collapse at the same critical point $\alpha = 45°$, as predicted by equations (17) and (26) when $d \rightarrow \infty$ and shown in figure 4. This means that in this case the transition is only to a periodic director field. It is interesting that, although $\alpha = 45°$ is out of the range of values of α of validity for the linear theory, as discussed above, this explains results from NMR magnetic reorientation experiments with large samples, where the onset of a periodic distortion is at angles $\alpha_c \cong 45°$ [3, 5, 6].

4. Conclusions

The linear theory of the magnetic field induced instabilities in the reorientation of the nematic director in non-Fréedericksz geometries ($\mathbf{H}$ not normal to $\mathbf{n}_0$) predicts that when the control parameter is the angle of the magnetic field to the initial homogeneous director field, two transitions from a homogeneous to a distorted director field reorientation are possible.

For bounded samples between two parallel plates, the first transition, to a non-periodic deformed state, is at a critical angle given by $\cos 2\alpha_{c1} = -(H^*/H)^2$, where H^* is the critical Fréedericksz field. This transition corresponds to that determined by a static stability analysis [20, 21] which allowed it to be classified as the equivalent of a second order phase transition.

The second transition, at a critical angle $\alpha_{c2} > \alpha_{c1}$, to a periodic distorted state, depends also on several viscoelastic parameters, as shown by equation (26). This second transition is equivalent to a second order phase transition, where the wave vector corresponding to the maximum growth rate of the instability plays the role of the order parameter. The results show that the periodic mode is cut off at a higher reduced field $h = H/H^*$ when the angle α goes from 90° to lower values.

Critical sample thicknesses can be put in evidence for both the aperiodic and the periodic director reorientations for a given magnetic field. The first critical thickness, separating the homogeneous from the aperiodic deformed state, is proportional to the magnetic coherence length. This indicates that only the interplay between the elastic coupling to the surface and the orientation in the bulk is important in order to determine the transition to the aperiodic distorted state. The second critical thickness, separating the aperiodic from the periodic distorted state, has further contributions from viscosity and elastic parameters, as shown by equation (28).

The analytical solutions (26) or (28) allow the conclusion that the value of the ratios of the viscoelastic parameters therein indicates how much the second transition will depart from the first. This value is a measure of the compromise between the reduction of viscosity of the fastest growing modes (with short wavelengths and low viscosity) and the reduction of elastic energy of the slower modes (with long wavelengths and high viscosity, but energetically favoured).

The author wishes to thank Prof. A. F. Martins and Prof. J. Figueirinhas for helpful comments. The author also wishes to thank the referee for his detailed comments on the original manuscript, which helped to improve this paper. This work was partly financed by JNICT of Portugal under research contract PBIC/C/CEN/1049/92.

Appendix

Linear stability analysis of a twist-bend instability

The system (6, 8) consists of two first order, ordinary, linear differential equations for the amplitudes of the perturbations. It is a dynamical system that can be solved in a standard manner by the hypothesis

$$\begin{aligned} v_0(t) &= v_0(0) \exp(\lambda t) \\ \theta_0(t) &= \theta_0(0) \exp(\lambda t). \end{aligned} \tag{A1}$$

The local dynamical and structural stability properties at the transition point are determined by the eigenvalues of the stability matrix $F_{ij} \equiv \partial F_i / \partial j$, where the functions F_{ij}, $i, j \equiv v_0$, θ_0, are defined in equations (6) and (8). These eigenvalues are given by

$$\lambda_{\pm} = (\delta \pm \sqrt{\omega^2})/2 \tag{A2}$$

with

$$\delta = -\left(\frac{a}{\gamma_1} + \frac{c}{\rho}\right) \tag{A3}$$

where a is defined by equation (7) and

$$\omega^2 = \delta^2 - 4\Delta \tag{A4}$$

where Δ is the determinant of the stability matrix, given by

$$\Delta = \frac{a}{\rho\gamma_1}(\eta_a q_y^2 + \eta_c q_z^2). \tag{A5}$$

When both eigenvalues are real and unequal, the system is locally equivalent to a gradient system [22]. In this case there is a bifurcation when one or both eigenvalues assume the value zero, which implies a zero determinant of the stability matrix and defines the bifurcation set

$$\Delta = 0. \tag{A6}$$

For gradient-like systems with eigenvalues $(+,+)$, $(\pm, \mp)$ or $(-,-)$ the critical point is an unstable node, a saddle or a stable node, respectively. The resolution of equation (A6) yields for real wave vectors the critical point

$$a = 0. \tag{A7}$$

This result is the same equation as that found for the critical point in a static analysis with the wave vector as a free parameter [20, 21], which is consistent with the fact that for real eigenvalues the dynamical system (6, 8) is locally equivalent to a gradient system. In [20, 21] it was shown that this transition is the equivalent of a second order phase transition. As in the static case, the state with $a < 0$ is the unstable state.

In the case of real wave vectors, since the viscosities γ_1, η_a and η_b are positive, we see from (A2–A5) that a sufficient condition for real $\lambda_{\pm}$ is $a \leqslant 0$, that is to say at or above the transition. If $a > 0$ the eigenvalues may be complex. In this case the stability properties of the system (6, 8) are determined by the sign of the real part of the eigenvalues [22] that by equations (A3), (7) and (9) can be written

$$\delta = -b/\gamma_1 \tag{A8}$$

where

$$b = \left(1 + \frac{1}{\mu_{\text{twist}}}\right) K_2 q_y^2 + \left(1 + \frac{1}{\mu_{\text{bend}}}\right) K_3 q_z^2 + \chi_a H^2 \cos 2\alpha \tag{A9}$$

and where

$$\mu_{\text{twist}} = \frac{K_2/\gamma_1}{\eta_a/\rho} \tag{A10}$$

and

$$\mu_{\text{bend}} = \frac{K_3/\eta_{\text{bend}}}{\eta_c/\rho} \tag{A11}$$

are dimensionless quantities that may be interpreted as the ratio of two diffusion constants [1]. The critical point is a stable focus if $\delta < 0$ and an unstable focus if $\delta > 0$ [22], the bifurcation set being defined by

$$\delta = 0 \tag{A12}$$

which implies by (A8)

$$b = 0. \tag{A13}$$

When comparing the bifurcation sets given by equations (A7) and (A13), one can see that the transition points can be significantly different, because the additional terms in (A9) stem from the inverse of the quantity μ, which is usually very small (except in the case of high molecular mass polymer liquid crystals where it can be of the order of unity). For real wave vectors, c defined by equation (9) is always positive. Since before the transition defined by equation (A7) the system is in a state with $a > 0$, this means by (A3) that $\delta < 0$, which implies, according to (A12), that the unperturbed state is stable. We can then conclude that for real wave vectors the transition happens at $a = 0$. Finally, from equations (A6, A7) and (A12, A13) follows the conclusion that the inertial term in the velocity equation must be kept in order to get the transition (A13).

References

[1] De Gennes, P. G., and Prost, J., 1993, *The Physics of Liquid Crystals*, 2nd Edn (Oxford: Clarendon Press).

[2] Chandrasekhar, S., 1994, *Liquid Crystals*, 2nd Edn (Cambridge University Press).

[3] Filas, R. W., Hadjo, L. E., and Eringen, A. C., 1974, *J. chem. Phys.*, **61**, 3037; Orwoll, R. D., and Vold, R. L., 1971, *J. Am. chem. Soc.*, **93**, 5335; Filas, R. W., 1978, *Mesomorphic Polymers and Polymerization in Liquid Crystalline Media*, edited by A. Blumstein, ACS Symposium Series 74 (Washington DC: ACS), Chap. II.

[4] Hui, Y. W., Kuzma, M. R., San Miguel, M., and Labes, M. M., 1985, *J. chem. Phys.*, **83**, 288.

[5] Martins, A. F., Esnault, P., and Volino, F., 1986, *Phys. Rev. Lett.*, **57**, 1745.

[6] Esnault, P., Casquilho, J. P., Volino, F., Martins, A. F., and Blumstein, A., 1990, *Liq. Cryst.*, **7**, 607.
[7] Gotzig, H., Grunenberg-Hassanein, S., and Noack, F., 1994, *Z. Naturforsch.*, **49a**, 1179.
[8] Hughes, J. R., Luckhurst, G. R., and Picken, S. J., 1994, 15th International Liquid Crystal Conference.
[9] Guyon, E., Meyer, R., and Salan, J., 1979, *Mol. Cryst. liq. Cryst.*, **54**, 261.
[10] (*a*) Lonberg, F., Fraden, S., Hurd, A., and Meyer, R. B., 1984, *Phys. Rev. Lett.*, **52**, 1903; (*b*) Hurd, A. J., Fraden, S., Lonberg, F., and Meyer, R. B., 1985, *J. Physique*, **46**, 905.
[11] Kuzma, M., 1986, *Phys. Rev. Lett.*, **57**, 349.
[12] Srajer, G., Fraden, S., and Meyer, R. B., 1989, *Phys. Rev. A*, **39**, 4828.
[13] Rey, A. D., and Den, M. M., 1989, *Liq. Cryst.*, **4**, 409.
[14] Kini, U. D., 1991, *J. Phys. II Fr.*, **1**, 225.
[15] Buka, A., and Kramer, L., 1992, *J. Phys. II Fr.*, **2**, 315; Buka, A., and Kramer, L., 1992, *Phys. Rev. A*, **45**, 5624.
[16] Schwenk, N., and Spiess, H. W., 1993, *J. Phys. II Fr.*, **3**, 865.
[17] Brochard, F., Pieranski, P., and Guyon, E., 1972, *Phys. Rev. Lett.*, **28**, 1681; Pieranski, P., Brochard, F., and Guyon, E., 1973, *J. Physique*, **34**, 35.
[18] Karn, A. J., Shen, Y. R., and Santamato, E., 1990, *Phys. Rev. A*, **41**, 4510.
[19] Kini, U. D., 1991, *Liq. Cryst.*, **10**, 597; Kini, U. D., 1992, *Liq. Cryst.*, **12**, 449.
[20] Casquilho, J. P., Gonçalves, L. N., and Martins, A. F., 1996, *Liq. Cryst.*, **21**, 651.
[21] Casquilho, J. P., 1998, *Liq. Cryst.*, **24**, 549.
[22] Gilmore, R., 1993, *Catastrophe Theory for Scientists and Engineers* (Dover).
[23] Huang, K., 1987, *Statistical Mechanics*, 2nd Edn (Wiley).
[24] Haken, H., 1983, *Synergetics*, 3rd Edn (Springer).